Technical Notes

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Numerical Solution of the Equation for a Thin Airfoil in Ground Effect

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I. Introduction

THE literature on ground effect in the theory of thin airfoils is extremely rich; presenting it here is not necessary. However, we refer to Refs. 1-4, where one may find the thin-airfoil equation for incompressible flow. This equation cannot be integrated exactly. Several papers[3-5] provide methods of successive approximations for the solution. This Note attempts an integration of the thin-airfoil equation in ground effect with Gauss-type quadrature formulas.

II. Integral Equation

Before presenting the solution of the integral equation, we provide a method for obtaining it. This method differs from other existing ones and may be applied to the general case of compressible fluids in the presence of thin airfoils. Unlike the classical methods where the thin airfoil is replaced by a vortex distribution, in this method the airfoil is replaced by a force distribution, and the intensity of this distribution is determined such that its effect on the fluid is the same as the effect of the wing. The method has been applied in Refs. 6 and 7. In order to satisfy the condition of vanishing for the Oy component of the velocity on the ground, it is also considered a symmetric distribution with respect to the ground.

Assume that a uniform motion of velocity U_∞, pressure p_∞, and density ρ_∞ of an ideal fluid is perturbed by the presence of a thin airfoil. Using dimensionless variables, the reference length L_0 being half of the airfoil chord and choosing the Ox axis in the direction of the freestream flow, for the total velocity V_1 and total pressure P_1 we put

$$V_1 = U_\infty(i + v), \qquad P_1 = p_\infty + \rho_\infty U^2{}_\infty p \tag{1}$$

We denote

$$y = h(x) \pm h_1(x), \qquad |x| < 1 \tag{2}$$

where

$$h(x) = -\epsilon\bar{h}(x), \qquad h_1(x) = -\epsilon\bar{h}_1(x) \tag{2'}$$

are the airfoil equations (the origin O is taken at the middle of the chord), with ϵ being a small parameter (for instance, it may be the incidence). It is shown[6] that in the linear approximation with respect to ϵ, the perturbation is determined by the system

$$M^2\frac{\partial p}{\partial x} + \operatorname{div} v = 0, \qquad \frac{\partial v}{\partial x} + \operatorname{grad} p = 0 \tag{3}$$

and its fundamental solution is

$$p(x, y) = \frac{1}{2\pi\beta}\frac{x_0 f_1 + \beta^2 y_0 f}{x_0^2 + \beta^2 y_0^2}, \qquad v(x, y) = \frac{\beta}{2\pi}\frac{x_0 f - y_0 f_1}{x_0^2 + \beta^2 y_0^2} \tag{4}$$

where M is the Mach number in freestream flow and

$$x_0 = x - \xi, \qquad y_0 = y - \eta, \qquad f = (f_1, f), \qquad v = (u, v) \tag{4'}$$

and where $\beta = (1 - M^2)^{1/2}$. The meaning of this solution is that it gives the perturbation produced in the foregoing uniform flow by the force f applied at the point (ξ, η).

Replacing the wing with a force distribution of unknown intensity (f_1, f) (ξ) and the image wing symmetric with respect to $y = -D$ by a distribution of intensity $(f_1, -f)$ (ξ) leads to the following representation:

$$\begin{aligned} p(x, y) &= \frac{1}{2\pi\beta}\int_{-1}^{+1}\frac{x_0 f_1(\xi) + \beta^2 y f(\xi)}{x_0^2 + \beta^2 y^2}\,\mathrm{d}\xi \\ &+ \frac{1}{2\pi\beta}\int_{-1}^{+1}\frac{x_0 f_1(\xi) - \beta^2(y + 2D) f(\xi)}{x_0^2 + \beta^2(y + 2D)^2}\,\mathrm{d}\xi \\ v(x, y) &= \frac{\beta}{2\pi}\int_{-1}^{+1}\frac{x_0 f(\xi) - y f_1(\xi)}{x_0^2 + \beta^2 y^2}\,\mathrm{d}\xi \\ &- \frac{\beta}{2\pi}\int_{-1}^{+1}\frac{x_0 f(\xi) + (y + 2D) f_1(\xi)}{x_0^2 + \beta^2(y + 2D)^2}\,\mathrm{d}\xi \end{aligned} \tag{5}$$

Taking the limit as $y \to \pm 0$, we obtain

$$f(x) = p(x, +0) - p(x, -0) \tag{6}$$

$$\begin{aligned} v(x, \pm 0) &= \mp\frac{1}{2}f_1(x) \\ &+ \frac{\beta'}{2\pi}\int_{-1}^{+1}\frac{f(\xi)}{x - \xi}\,\mathrm{d}\xi - \frac{\beta}{2\pi}\int_{-1}^{+1}\frac{x_0 f(\xi) + 2D f_1(\xi)}{x^2 + 4\beta^2 D^2}\,\mathrm{d}\xi \end{aligned} \tag{7}$$

The prime indicates the principal value in the Cauchy sense.

Imposing the boundary conditions

$$v(x, \pm 0) = h'(x) \pm h_1'(x), \qquad |x| < 1 \tag{8}$$

we obtain

$$f_1(x) = -2h_1'(x) \tag{9}$$

$$\frac{\beta'}{2\pi}\int_{-1}^{+1}\frac{f(\xi)}{x - \xi}\,\mathrm{d}\xi - \frac{\beta}{2\pi}\int_{-1}^{+1}\frac{x_0 f(\xi)}{x_0^2 + 4\beta^2 D^2}\,\mathrm{d}\xi = H(x) \tag{10}$$

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where taking into account (2′), we get $H(x) = -\epsilon \bar{H}(x)$ and

$$\bar{H}(x) = \bar{h}'(x) - \frac{2\beta D}{\pi} \int_{-1}^{+1} \frac{\bar{h}'_1(\xi)}{x_0^2 + 4\beta^2 D^2} \, d\xi \tag{10'}$$

The condition $v(x, -D) = 0$ may be easily checked. The integral equation of the problem is Eq. (10). For the incompressible fluid ($\beta = 1$) and for a zero thickness airfoil ($h_1 = 0$), it coincides with the equation given in Refs. 2 and 4. For a symmetrical airfoil ($h = 0$), this equation is just Eq. (10) from Ref. 3. In Plotkin and Kennel's theory,[3] $\gamma(x)$ stands for $p(x, -0) - p(x, +0)$.

III. Solution of the Integral Equation

It is known[8] that the solution of an integral equation of the form of Eq. (10) depends on the behavior imposed at the points ± 1. In aerodynamics, one imposes the solution satisfying the Kutta-Joukowski condition at the trailing edge. This kind of solution has the form

$$\beta f(\xi) = -\epsilon \left(\frac{1-\xi}{1+\xi} \right)^{1/2} F(\xi) \tag{11}$$

Applying the general theorem given in Ref. 9, we deduce the following Gauss-type quadrature formulas:

$$\frac{1}{2\pi} \int_{-1}^{+1} \left(\frac{1-\xi}{1+\xi} \right)^{1/2} F(\xi) \, d\xi = \frac{1}{2n+1} \sum_{\alpha=1}^{n} (1 - \xi_\alpha) F(\xi_\alpha)$$

$$\frac{1'}{2\pi} \int_{-1}^{+1} \left(\frac{1-\xi}{1+\xi} \right)^{1/2} \frac{F(\xi)}{\xi - x_i} \, d\xi = \frac{1}{2n+1} \sum_{\alpha=1}^{n} \frac{1-\xi_\alpha}{\xi_\alpha - x_i} F(\xi_\alpha) \tag{12}$$

$$x_i = \cos \frac{2i-1}{2n+1} \pi, \qquad \xi_\alpha = \cos \frac{2\alpha\pi}{2n+1} \tag{12'}$$

with $i = 1, \ldots, n$, $\alpha = 1, \ldots, n$. These formulas are exact for an F polynomial of the $(2n-1)$th degree. Choosing a sufficiently large n, we can use these formulas for every F.

With Eqs. (11) and (12), the integral equation (10) reduces to the following algebraic system:

$$\sum_{\alpha=1}^{n} (A_{i\alpha} + B_{i\alpha}) F_\alpha = \bar{H}_i, \qquad i = 1, \ldots, n \tag{13}$$

where

$$A_{i\alpha} = \frac{1}{2n+1} \frac{1-\xi_\alpha}{x_i - \xi_\alpha}$$

$$B_{i\alpha} = -\frac{1}{2n+1} \frac{(1-\xi_\alpha)(x_i - \xi_\alpha)}{(x_i - \xi_\alpha)^2 + 4\beta^2 D^2} \tag{13'}$$

with the notation $F_\alpha = F(\xi_\alpha)$, $\bar{H}_i = \bar{H}(x_i)$.

From Eq. (13), the values of $F_1, \ldots, F_n$ are determined. Once these are obtained, the lift and moment coefficients are given by the formulas

$$c_L = \int_{-1}^{+1} f(\xi) \, d\xi = \frac{\epsilon}{\beta} \int_{-1}^{+1} \left(\frac{1-\xi}{1+\xi} \right)^{1/2} F(\xi) \, d\xi = \frac{2\pi\epsilon}{\beta} N_1$$

$$c_M = -\frac{1}{2} \int_{-1}^{+1} \xi f(\xi) \, d\xi = -\frac{\pi\epsilon}{2\beta} N_2 \tag{14}$$

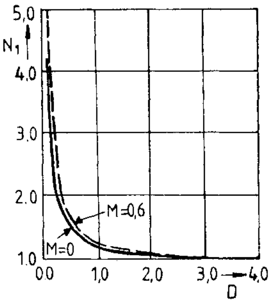

Fig. 1 The N_1 for a flat plate.

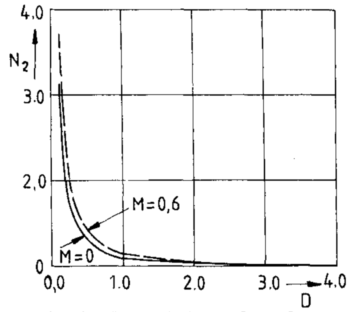

Fig. 2 The N_2 for a flat plate.

where

$$N_1 = \frac{1}{2n+1} \sum_{\alpha=1}^{n} (1 - \xi_\alpha) F_\alpha$$

$$N_2 = -\frac{2}{2n+1} \sum_{\alpha=1}^{n} \xi_\alpha (1 - \xi_\alpha) F_\alpha \tag{14'}$$

Equation (13) is programmed for the computer. It is seen that $A_{i\alpha}$ and $B_{i\alpha}$ do not depend on the form of the airfoil so that the program still holds for any airfoil; the only thing that is changed is the column matrix $\bar{H}_i$.

IV. Numerical Results

For a flat plate at ϵ incidence ($h = -\epsilon x$, $h_1 = 0$), we have $\bar{H}_i = 1$; c_L and c_M are of the form

$$c_L = c_L^\infty N_1, \qquad c_M = c_M^\infty N_2 \tag{15}$$

where c_L^∞ and c_M^∞ are the coefficients in the case of aerodynamics in the absence of ground effect ($D = \infty$). The dependence of N_1 and N_2 on D is given in Figs. 1 and 2. The compressibility effect is larger than shown in the figures since β is involved also in c_L^∞ and c_M^∞. The ground influence is very large for small D, and it disappears quickly as D is increased.

As it is known, in the ground absence, if the profile is symmetrical and at 0 deg incidence, the lift vanishes. This does not apply in the case of ground presence.[3] For the symmetrical Joukowski airfoil considered in Ref. 3

$$\bar{h}_1 = -(1-x)(1-x^2)^{1/2} \tag{16}$$

with Gauss-type quadrature formulas

$$\frac{1}{\pi} \int_{-1}^{+1} (1-\zeta^2)^{1/2} G(\zeta) \, d\zeta = \frac{1}{m+1} \sum_{\gamma=1}^{m} (1 - \zeta_\gamma^2) G(\zeta_\gamma) \tag{17}$$

where

$$\zeta_\gamma = \cos \frac{\gamma\pi}{m+1}, \qquad \gamma = 1, \ldots, m \tag{17'}$$

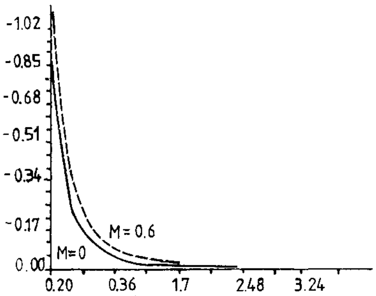

Fig. 3 The N_1 for a symmetrical Joukowski airfoil.

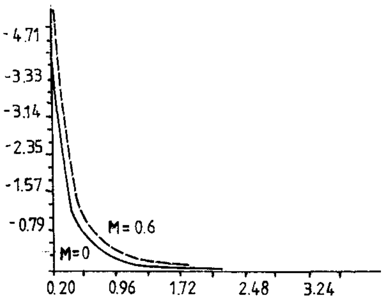

Fig. 4 The N_2 for a symmetrical Joukowski airfoil.

we obtain

$$\bar{H}_i = \frac{2\beta D}{m+1}\sum_{\gamma=1}^{m}\frac{1-\zeta_\gamma^2}{(x_i-\zeta_\gamma)^2+4\beta^2D^2} - \frac{4\beta D}{2m+1}\sum_{\alpha=1}\frac{(1-\zeta_\gamma)\zeta_\gamma}{(x_i-\zeta_\alpha)^2+4\beta^2D^2} \tag{18}$$

In this case, the dependence of N_1 and N_2 on D is given in Figs. 3 and 4. The lift is negative, i.e., the resulting force is toward the ground.

The present theory, linear with respect to the thickness parameter ϵ, is not valid in the close neighborhood of the ground where D becomes a small parameter too. In this region, a new theory is necessary.

Acknowledgment

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Optimum Hypersonic Airfoil with Power Law Shock Waves

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Introduction

IN the present Note, the flowfield over a class of two-dimensional lifting surfaces is examined from the viewpoint of inviscid, hypersonic small-disturbance theory (HSDT). It is well known that a flowfield in which the shock shape $S(x)$ is similar to the body shape $F(x)$ is only possible for $F(x)=x^k$ and the freestream Mach number $M_\infty=\infty$. This self-similar flow has been studied for several decades since it represents one of the few existing exact solutions of the equations of HSDT. Detailed discussions are found, for example, in papers by Cole,[1] Mirels,[2] Chernyi,[3] and Gersten and Nicolai,[4] but they are limited to convex body shapes, that is, $k \le 1$. The only study of concave body shapes was attempted by Sullivan[5] where only special cases were considered. The method used here shows that similarity also exists for concave shapes, and a complete solution of the flowfield for any $k > ⅔$ is given. The effect of varying k on $C_L^{3/2}/C_D$ is then determined, and an optimum shape is found. Furthermore, a wider class of lifting surfaces is constructed using the streamlines of the basic flowfield and analysed with respect to the effect on $C_L^{3/2}/C_D$.

We neglect viscous effects and assume boundary layers to be thin and attached to the surface. The surfaces are considered to correspond to the lower compression surface of a two-dimensional wing. Since the pressure difference across the shock induced by this surface is of higher order than that of the shock induced by the upper expansion surface, we neglect the contribution of the upper surface to the lift or drag.

Similarity Solution

This section is a formulation of our problem in the framework of hypersonic small-disturbance theory. If we substitute the scaled variables $y=\bar{y}/\delta$ and $x=\bar{x}$, with δ = thickness ratio, together with the asymptotic representations for velocity, pressure, and density into the equations of motion and neglect $O(\delta^2)$ terms, we obtain a reduced problem with the longitudinal momentum equation uncoupled from the rest of the problem. This longitudinal momentum equation can later be determined using the Bernoulli equation.

For a slender airfoil we write for the body surface $\bar{y}=\delta F(x)$ with associated shock shape $\bar{y}=\delta S(x)$. See Fig. 1.

Next, we change the (x,y) coordinate system to the (x,ψ) coordinate system, where ψ is the stream function. See Fig. 2. We further change from (x,ψ) to (x,ξ) coordinates, where ξ is the shock location $x=\xi$, i.e., the x location where an incoming streamline crosses the shock. See Figs. 2 and 3. Note that ψ and ξ are related by $\theta(\xi)\partial/\partial\psi=\partial/\partial\xi$, where $\theta(\xi)=\mathrm{d}S(\xi)/\mathrm{d}\xi$ can be determined by a separation of variables argument together with the continuity and momentum equation as $\theta(\xi)=ka\xi^{k-1}$. Also note that now the density can be written in terms of the pressure using the entropy equation and the shock conditions. Finally, we obtain for our basic problem

Continuity:

$$\frac{c^2\theta(\xi)^{\frac{2}{\gamma}+1}}{p^{*\frac{1}{\gamma}}}\frac{\partial p^*}{\partial x}+\frac{\partial v^*}{\partial \xi}=0 \tag{1}$$

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